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Investigation of the behaviour of a column beyond the elastic limit by methods of the technical theory of stability☆

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Abstract

The behaviour of a model column on a deformable base acted upon by compressive and lateral loads beyond the elastic limit is investigated using the methods of the technical theory of stability. The loading trajectory which leads to the greatest deflection is constructed for specified constraints on the absolute magnitude of the lateral load and a monotonic increase in the compressive load, and the possibility of the occurrence of a limiting state is investigated.

There are two approaches, the Karman approach[1] and the Shanley approach,[2] for describing the stability of a compressed column beyond the elastic limit, which lead to different values of the critical loads, called the reduced modulus load and the tangent modulus load respectively. In the case of a rectilinear column, the essence of these approaches is as follows: the condition for the occurrence of other equilibrium states, differing from the rectilinear equilibrium state in the case of a constant compressive force, serves as the criterion which leads to the first type of critical load, while the condition for the possibility of the occurrence of distortion when the compressive load is increased serves as the criterion leading to the second type of critical load.

The idea of investigating the roles of each of these critical loads without the direct use of the concept of stability, proposed for the first time by A. A. Il'yushin and V. A. Lomakin, rests on a concept of the technical theory of stability and is as follows. The magnitude of the change in the compressive load and the maximum value of the transverse perturbations acting on a given column can be estimated under real conditions, and the greatest possible deflection of the column can be estimated on the basis of this. For instance, a column under the action of a compressive load P and a lateral perturbing load N is said to be stable in the sense of technical stability if, for any loading processes $P(t)$, $N(t)$ (t is a loading parameter) for which $0 < P(t) \le P_m$, $|N(t)| \le N_0$(P_m and N_0 are known quantities), the greatest possible deflection does not exceed a specified value ε.

One of the techniques for investigating the problem of technical stability involves the construction of the extremal loading trajectory, in which the column attains the greatest deflection, and its subsequent investigation. Note that the determination of the maximum deflection in the above-mentioned formulation is not only of theoretical interest but also of practical interest in the case of a real structure. Extremal trajectories have been constructed previously[3] in the case of a model Shanley column and an idealized I-beam. The maximum lateral load is initially applied on the

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trajectory for these models and the compressive force is then increased up to the maximum value. In the geometrically linear formulation, it is found that the magnitude of the maximum deflection is finite as long as the compressive load is less than the reduced modulus load and tends to infinity as the compressive load tends to the reduced modulus load. However, the results obtained for a model column cannot be transferred to the real analogue due to the fact that there is no central layer in the case of the model column. The process of the removal of the lateral load from it with its subsequent application is therefore described in the majority of cases by an elastic law which leads to the loss of the plastic effect of the accumulation of deflection.

A model of a column on the deformable layer is considered; it is similar to that proposed by Klyushnikov,[4] which takes account of the effect of the plastic accumulation of deflections.

Note that, in this case, the corresponding criteria for loss of stability are obtained[4] under the assumption that the compressive load exceeds the plastic load, that is, a cross-section of the column is found to be in a plastic state at the instant of loss of stability. The method of elastoplastic conditioning[5,6] enables one to increase the plastic load up to a value which exceeds the reduced modulus load. However, when account is taken of the effect of the lateral perturbing load, a limiting state is possible for which there are no quasistatic equilibrium states when certain loading process parameters are increased.

It is assumed that the maximum value of the non-decreasing compressive load exceeds the reduced modulus load but is less than the corresponding plastic load. The problem of finding the greatest deflection is solved and the question concerning the possibility of reaching a limiting state is also investigated under the assumption that the maximum value of the lateral load is known. With reference to the properties of the column material, the hypothesis of linear hardening with elastic unloading is adopted without taking account of the effect of secondary plastic deformations.

For convenience in writing out the relations between the stresses and strains, we will introduce the concept of minimum strain

$$e_m : e_m = \min_{0 \le \tau \le t} \{e(\tau), -\varepsilon_s\}$$

The equation relating σ and e then has the form

$$\sigma = -E\varepsilon_s + E_1(e_m + \varepsilon_s) + E(e - e_m) \tag{1}$$

where E is Young's modulus, E_1 is the modulus of strain hardening, e is the strain, e_m are the minimum strains and ε_s is the elastic limit (the direction of elastic unloading is shown by the arrow in Fig. 1).

A sketch of the model column is shown in Fig. 2. The following notation is used: $2l$ is the length of the column ($OO_1 = O'O'_1 = l$), φ is the angle of inclination, $2h$, $2s$, b are the initial dimensions of the deformable layer ABB_1A_1, $2h = AB$ is the width of the layer, $2s$ is the initial height and b is the size of the layer in a direction perpendicular to the plane of the diagram.

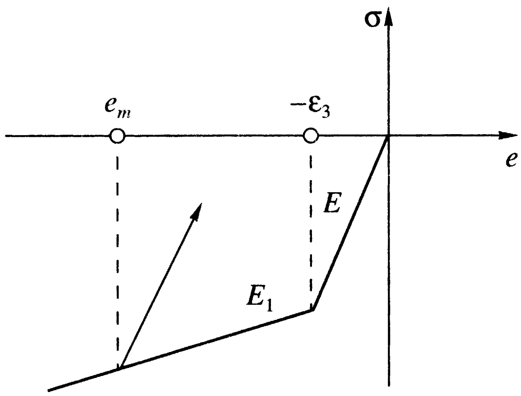

Fig. 1.

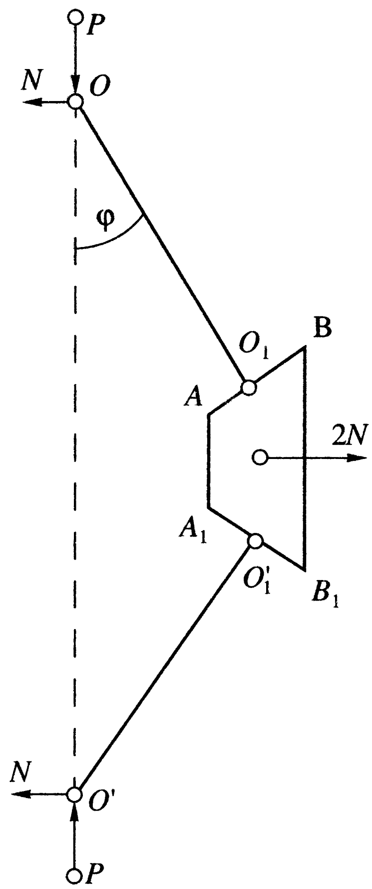

Fig. 2.

Taking account of expression (1), we can write the equations for the equilibrium of the column on the deformable layer in the geometrically linear formulation in the form

$$2bhEe(0) = -P + 2bh\Delta E\varepsilon_s + b\Delta E\int_{-h}^{h} e_m(z)dz$$
$$\left(\frac{2bh^3E}{3s} - Pl\right)\varphi = Nl + b\Delta E\int_{-h}^{h} e_m(z)zdz \tag{2}$$

where $\Delta E = E - E_1$, $e(z)$ and $e_m(z)$ are the distribution of the strains and the minimum strains in a cross-section of the column (the origin of the z axis is located at the centre of the deformable layer).

According to the hypothesis of plane sections, we have

$$e(z) = e(0) + \varphi z/s \tag{3}$$

In the case of non-decreasing P, we obtain from Eq. (2) that $e(0)$ is a non-increasing function of the loading parameter. It follows from condition (3) that $e(z)$ is a convex function of z (note that one can put $e_m(z) = -\varepsilon_s$ at the initial instant).

We will now formulate the criterion for a limiting state of the column. In the model being considered, the critical forces are: the Euler force P_l, the reduced modulus force P_k, and the tangent modulus force P_t which are respectively equal to[4]

$$P_l = 2bh^3E/(3sl),\quad P_k = 4\alpha^2P_l/(1+\alpha)^2,\quad P_t = \alpha^2P_l,\quad P_s = 2bhE\varepsilon_s;\quad \alpha^2 = E_1/E$$

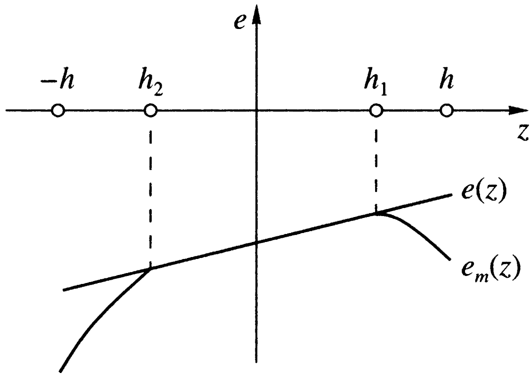

Fig. 3.

In the most general case, there are three zones in the cross-section of a column (Fig. 3): (1) $-h<z<h_2$, where $e_m(z)<e(z)$; (2) $h_2<z<h$, where $e_m(z)=e(z)$; (3) $h_1<z<h$, where $e_m(z)<e(z)$. In the case of the first and third zones, the relation between the stress increments and the strain increments is determined by the elastic law, and, in the case of the second zone, it depends on the direction in which the strain changes. To formulate the criteria of the limiting state, we introduce the function

$$G(h_1, h_2) = \frac{b}{12s}\{4[2h^3E-(h_1^3-h_2^3)\Delta E]-3(h_1^2-h_2^2)^2\Delta E^2/[2hE-(h_1-h_2)\Delta E]\}$$

Note that $\partial G/\partial h_1 \leq 0$, $\partial G/\partial h_2 \geq 0$.

The criteria for a limiting state are formulated in Assertions 1–3 presented below (a scheme for their proof was presented earlier in Ref. 7).

Assertion 1. If $G(h_1, h_2)-P_l>0$, then any sufficiently small quasistatic continuations of the loading process $\Delta P(t)$, $\Delta N(t)$ are possible.

Assertion 2. If

$$G(h_1, h_2)-Pl<0, \quad G(h^+, h_2)-Pl>0, \quad G(h_1, h^-)-Pl>0$$

where $h^{\pm}\in[h_2, h_1]$ is a root of the corresponding equation $\pm 4hh^{\pm}E-(h^{\pm}-h_i)^2\Delta E=0$ ($i=1$ in the case when h^+ and $i=2$ in the case when h^-), then any sufficiently small continuations of the loading process $\Delta P(t)$, $\Delta N(t)$ are possible.

Assertion 3. If $G(x, h_2)-Pl<0$, where $x=h^+$ and if $h^+\in[h_2, h_1]$ and $x=h_1$ otherwise, then quasistatic equilibrium states exist which are characterized by an unlimited increase in the deflection of the column in the case of a constant compressive force and a non-increasing lateral force.

Assertions 1–3 have a simple mechanical meaning: if quasistatic equilibrium states exist in the case of a simple increase in the lateral load and the deflection becomes larger at the same time, then any sufficiently small continuations of the loading process are possible.

Two cases are considered for constructing the extremal loading trajectory and finding the maximum deflection.

The case when $G(0, -h)-P_ml<0$ (sufficiently large values of the compressive force). Here, P_m is the largest possible value of the compressive force. Then,

$$P_m > bh^3(E^2+14EE_1+E_1^2)/(12lsQ_4) = P_m^*, \quad Q_4 = E+E_1$$

Suppose $P_m=G(y, -h)/l$, $y\in(-h, 0)$. We shall show that the extremal deflection is attained by applying the maximum lateral load N_m to a column which is compressed by a force P_m. Suppose φ_m is the corresponding angle of inclination. For sufficiently large values of N_m, we have the following distribution of the minimum strains $e_m(z)$ (the dashed line

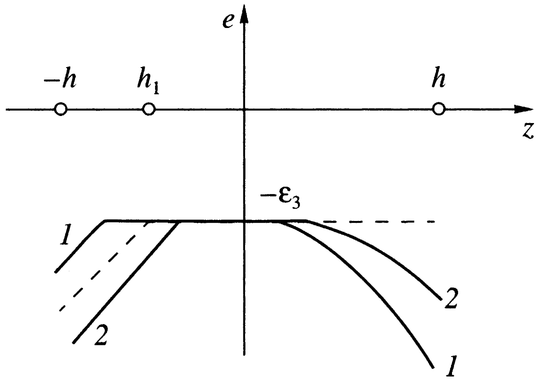

Fig. 4.

in Fig. 4)

$$e_m(z) = \begin{cases} e(z), & z \in [-h, h_1] \text{ (plastic zone)} \\ -\varepsilon_s < e(z), & z \in [h_1, h] \text{ (elastic zone)} \end{cases}$$

$$e(z) = -\varepsilon_s + \varphi_m(z - h_1)/s \qquad (4)$$

Using the equilibrium Eqs. (2) and (4), we obtain the following system of equations for determining φ_m and h_1 as functions of N_m for a specified value of P_m

$$\varphi_m = 2s\Delta P/(b(Q - Q_1)), \quad N_m l = \Delta P_s f(h_1, P_m) \qquad (5)$$

where

$$f(h_1, P_m) = \{b[(2h + h_1)Q + (2h - h_1)Q_1] - GP_m ls\}/(3b(Q - Q_1))$$

$$\Delta P_s = P_s - P_m, \quad Q = (h - h_1)^2 E, \quad Q_1 = (h + h_1)^2 E_1$$

From the condition $h_1 > -h$, we obtain that plastic deformation occurs when

$$N_m l > \Delta P_s (2bh^3 Q_4 - 6P_m ls)/(3bh^2 \Delta E)$$

The system of Eq. (5) can be considered as a parametric representation (parameter h_1) of φ_m and N_m.
Since

$$\partial f/\partial h_1 = 4sQ_2[G(h_1, -h) - P_m l]/(b(Q - Q_1))$$

where

$$Q_2 = (h - h_1)E + (h + h_1)E_1$$

we have

$$dN_m/dh_1 > 0 \text{ when } h_1 \in [-h, y], \quad dN_m/dh_1 < 0 \text{ when } h_1 \in [y, h]$$

Assertion 4. If $N_m l = \Delta P_s f(h_1, P_m)$, $h_1 \in [-h, y)$, then, for any loading processes $P(t)$, $N(t)$ from the class being considered, the angle of inclination of the column $\varphi(t)$ satisfies the condition

$$|\varphi(t)| \le \varphi_m, \quad \varphi_m = 2s\Delta P_s/(b(Q - Q_1))$$

Proof. Consider the class of loading processes $P(t)$, $N(t)$ which satisfies the conditions

$$dP/dt \ge 0, \quad P(t) \in [0, P_m], \quad |N(t)| \le N_{m1}, \quad N_{m1} < N_m$$

We will show by reductio ad absurdum that $\varphi(t) < \varphi_m$. Suppose a $t = t_1$ is found such that $P_1 = P(t_1) \le P_m$, $N_1 = N(t_1) < N_m$, $\varphi(t_1) = \varphi_m$ and, at the same time, $\varphi(\tau) < \varphi_m$, $\tau \in [0, t_1]$. Now, suppose that, when $t = t_1$, the distribution of the minimum strains throughout the cross-section of the column is specified by the function $e_m(z, t_1) = e_{m1}(z)$. We denote the distribution of the minimum strains, defined by formulae (4), by $e_{m0}(z)$. □

Two versions are possible.

Version 1. $e_{m1}(z, h_1) = -\varepsilon_s$ (curve 1 in Fig. 4). Then,

$$e_{m0}(z) \le e_{m1}(z) \text{ when } z \in [-h, 0], \quad e_{m0}(z) \ge e_{m1}(z) \text{ when } z \in [0, h]$$

We put

$$P_m - P_1 = \Delta P, \quad N_m - N_1 = \Delta N, \quad e_{m0}(z) - e_{m1}(z) = \Delta e_m(z), \quad I(a, b) = \int_a^b \Delta e_m(z) z dz$$

Note that $\operatorname{sign}\Delta e_m(z) = \operatorname{sign} z$.

From Eq. (2), we have

$$-\Delta P l \varphi_m = \Delta N l + b\Delta E I(-h, h)$$

Since $\Delta P \ge 0$, $\Delta N > 0$, $I(-h, h) > 0$, we have obtained a contradiction.

Version 2. $e_{m1}(z, h_1) < -\varepsilon$ (curve 2 in Fig. 4).

We denote the difference in the strains at the point $z = 0$ by $\Delta e = e_0(0) - e_1(0)$. Note that $\Delta e_m(z) = \Delta e$ when $z \in [-h, h_1]$ and $\Delta e_m(z)$ is a concave function in the interval $[h_1, h]$, which increases in the interval $[0, h]$.

From the first equation of system (2), we have

$$bQ_2\Delta e = b\Delta E \int_{h_1}^{h} \Delta e_m(z) dz - \Delta P \tag{6}$$

From relation (6), it follows that

$$\int_{h_1}^{h} \Delta e_m(z) dz > \Delta e(h - h_1) \tag{7}$$

It is easy to show that $\Delta e_m(z) \le \Delta e$ when $z \in [h_1, 0]$. Then,

$$\int_{h_1}^{0} \Delta e_m(z) z dz \ge -\Delta e h_1^2/2$$

Since $\Delta e_m(z)$ is a concave function and $\Delta e_m(0) \le \Delta e$, it then follows from relation (7) that $I(0, h) \ge \Delta e h^2/2$. Then, from the second equation of system (2), we obtain

$$-\Delta P \varphi_m l = \Delta N l + b\Delta E I(-h, h)$$

On transforming this equation taking account of the estimates which have been obtained, we have

$$- P\varphi_m l - \Delta N l = b\Delta E[I(-h, h_1) + I(h_1, 0) + I(0, h)] \ge$$
$$\ge b\Delta E[(h_1^2 - h^2)\Delta e/2 - h_1^2 \Delta e/2 + h^2 \Delta e/2] = 0$$

A contradiction has been obtained since $\Delta P \ge 0$, $\Delta N > 0$.

Letting N_{m1} tend to N_m, we arrive at the conclusion that, for any loading trajectory from the class being considered, $\varphi(t) \le \varphi_m$ and, consequently, by virtue of symmetry, $\varphi(t) \le \varphi_m$.

Assertion 5. If $N_m l = \Delta P_s f(h_1, P_m)$, where $h_1 \in [-h, y)$, the attainment of a limiting state for any loading processes from the class being considered is impossible.

The proof of this assertion follows from Assertion 3, since an unlimited increase in the angle of inclination is possible in the case of a reduction in the lateral load.

The case $G(0, -h) - P_m l > 0$. Suppose

$$P_m = G(y, -h)/l, \quad y \in (0, h_k), \quad h_k = h(1-\alpha)/(1+\alpha)$$

Assertion 6. Suppose $N_m l = \Delta P_s g(h_1, P_m)$, where

$$g(h_1, P_m) = f(h_1, P_m) \times (P_l - P_m)/[\Delta E b h^2 f(h_1, P_m)/(sl) + P_m - P_t]$$
$$h_1 \in [0, y], \quad \Delta P_s = (P_s - P_m)$$

Then, the angle of inclination of the column φ does not exceed φ_m for any loading trajectory.

Assertion 7. The magnitude φ_m of the angle of inclination of the column is the exact upper limit of the possible deflections, that is, loading trajectories exist which lead to an angle of inclination which is as close as desired to φ_m.

Assertion 8. If $N_m l = \Delta P_s g(h_1, P_m)$, $h_1 \in [0, y)$, then the attainment of a limiting state for any loading trajectory from the class being considered is impossible. If $N_m l > \Delta P_s g(y, P_m)$, then the attainment of a limiting state is possible.

We shall omit the proofs of Assertions 6–8.

Conclusions. In the case when the plastic load exceeds the critical Karman force, a limiting state can be attained which is characterized by the absence of quasistatic continuations of the loading process as the load is increased.

For sufficiently large compressive stresses, the maximum deflection is attained by applying a lateral load to the compressed column. A minimum value of the limiting lateral load is attained in the same loading trajectory.

At smaller values of the compressive load, but values which exceed the critical Karman force, the greatest deflection is attained under two sets of conditions: for a sufficiently small value of the lateral load, the maximum deflection is attained by applying a lateral load to the compressed column and, at high values of the lateral load, the mode of loading of a compressed column, in which the lateral load repeatedly changes sign according to a special law and accumulation of deflection occurs, corresponds to the maximum deflection.

An analogous loading trajectory brings the column into the limiting state for the minimum value of the lateral load.

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